銘傳大學 96 學年度研究所碩士班招生考試
資訊工程學系碩士班與資訊傳播工程學系碩士班
第二節

離散數學試題

（第 1 頁共 1 頁）
（限用答案本作答）

1. (20%).

   a. Show that $n^3 + (n+1)^3 + (n+2)^3$ is divided by 9, for n=1,2,....

   b. Show that $3^n + 7^n - 2$ is divided by 8, for n=1,2,....

2. (10%). Find the number of integer solutions of $x_1 + x_2 + x_3 = 15$ subject to $0 \le x_1 \le 6$, $x_2 \ge 0, x_3 \ge 0$.

3. (10%). Find the number of ways that two integers can be selected from $1, 2, \ldots, n-1$ so that their sum is greater than n.

4. (20%). Let $\mathbb{Z}_n$ denote the set of integers $\{0, 1, 2, \ldots, n-1\}$. Let $\odot$ be binary operation on $\mathbb{Z}_n$ such that

   $$a \odot b = \text{the remainder of } ab \text{ divided by } n$$

   a. Construct the table for the operation $\odot$ for $n = 4$.

   b. Show that $\{\mathbb{Z}_n, \odot\}$ is a semigroup for any n.

5. (10%). The Fibonacci sequence is definded by the recurrence relation

   $$a_n = a_{n-1} + a_{n-2}, \quad n \ge 3,$$

   and initial conditions

   $$a_1 = 1, \quad a_2 = 2.$$

   Show that a solution to this recurrence realtion is

   $$a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}.$$

6. (10%). Let X be a random variable with probability mass function $p(x_i) = \Pr(X = x_i)$, $i = 1, 2, \ldots$. Show that for every $k > 0$,

   $$\Pr(|X - \mu| \ge k\sigma) \le \frac{1}{k^2},$$

   where $\mu = E[X] < \infty$ and $\sigma^2 = E[(X-\mu)^2] < \infty$ denote the expectation and variance of X, respectively.

7. (20%). A function $f(x)$ is said to be convex over an interval (a, b) if for every $x_1, x_2 \in (a, b)$ and $0 \le \lambda \le 1$,

   $$f(\lambda x_1 + (1-\lambda)x_2) \le \lambda f(x_1) + (1-\lambda) f(x_2).$$

   Let X be a random variable taking on values $\mathscr{X} = \{x_1, x_2, \ldots, x_n\}$. Let $p(x) = \Pr(X = x)$, $x \in \mathscr{X}$, be the probability mass function of X.

   a. Show that $E[f(X)] \ge f(E[X])$. (Jensen's inequality).

   b. Use a. to show the information inequalty: Let $p(x), q(x)$, $x \in \mathscr{X}$, be two probability mass functions. Then

   $$D(p\|q) \ge 0,$$

   where

   $$D(p\|q) = \sum_{x \in \mathscr{X}} p(x) \log \frac{p(x)}{q(x)}$$

   is the relative entropy or Kullback-Leibler distance between $p(x)$ and $q(x)$.